Application of Voronoi Weights in Monte Carlo Integration with Given Sampling Plan

M. Vořechovský[1)], **V. Sadílek**[1)], and **J. Eliáš**[1)]

[1)]Faculty of Civil Engineering, Brno University of Technology, Brno, Czech Republic, vorechovsky.m@fce.vutbr.cz

Keywords: *Voronoi Tessellation; Monte Carlo Integration; Weighted Statistical Estimators; Sampling Plan;*

Abstract

The paper considers a problem of Monte Carlo integration of function that feature random variables. It is assumed, that there exists a given sampling plan (a matrix of N_{sim} points in N_{var}-dimensional space). This sampling plan can be a result of crude Monte Carlo sampling strategy, or a Latin Hypercube Sample (either random or optimized via some criterion).

Consider a random vector $\mathbf{X}$ with joint probability density function $f_{\mathbf{X}}(\mathbf{x})$ and cumulative density function $F_{\mathbf{X}}(\mathbf{x})$. Consider also a deterministic function that features the random vector: $g(\mathbf{x})$. The problem to be solved is the estimation of the following integral that involves an additional transformation S of function g:

$$\mathrm{E}[S\left[g(\mathbf{X})\right]] = \int\limits_{-\infty}^{\infty} \dots \int\limits_{-\infty}^{\infty} S\left[g(\mathbf{x})\right] \mathrm{d}F_{\mathbf{X}}\left(\mathbf{x}\right) \tag{1}$$

where $\mathrm{d}F_{\boldsymbol{X}}\left(\mathbf{x}\right) = f_{\boldsymbol{X}}\left(\mathbf{x}\right) \cdot \mathrm{d}x_1\ \mathrm{d}x_2\ \cdots\ \mathrm{d}x_{N_{\text{var}}}$ is the infinitesimal probability and where the particular form of the function $S\left[g\left(\cdot\right)\right]$ depends on the transformation of interest. A typical application is the estimation of statistical moments of g. For example, to get the mean value of g denoted as $\mathrm{E}[g(\cdot)]$, one can simply consider $S\left[g()\right] = g()$. In standard Monte Carlo integration, it is assumed that all the sampling point have been selected with an equal sampling probability, i.e. $1/N_{\text{sim}}$. In such a case, the integration of a function g of a vector of random variables $\mathbf{X}$ over the domain of all random variables can be estimated as an average over $i = 1, \dots, N_{\text{sim}}$ points with equal weights:

$$\mathrm{E}[S\left[g(\mathbf{X})\right]] \approx \frac{1}{N_{\text{sim}}} \sum_{i=1}^{N_{\text{sim}}} S\left[g(\mathbf{x}_i)\right] \tag{2}$$

The application may be e.g. in estimation of statistical moments of a function g which is a frequently encountered problem.

The paper explores the possibility to improve the naive approach in Eq. (2) by considering unequal weights. These weights are obtained by transforming the points into sampling probabilities (points within a unit hypercube) and then by constructing the Voronoi tessellation around each point. The volumes of individual cells are then used as weights instead of equal weight $1/N_{\text{sim}}$. Supposedly, this approach could have been considered superior over Eq. (2) because it can remove inaccuracies stemming from clusters of sampling points.